UNIFORMLY LIPSCHITZIAN GROUP ACTIONS ON HYPERCONVEX SPACES

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ABSTRACT. Suppose that $\{T_a : a \in G\}$ is a group of uniformly L-Lipschitzian mappings with bounded orbits $\{T_a x : a \in G\}$ acting on a hyperconvex metric space M. We show that if $L < \sqrt{2}$, then the set of common fixed points $\operatorname{Fix} G$ is a nonempty Hölder continuous retract of M. As a consequence, it follows that all surjective isometries acting on a bounded hyperconvex space have a common fixed point. A fixed point theorem for L-Lipschitzian involutions and some generalizations to the case of λ-hyperconvex spaces are also given.

1. Introduction

A metric space (M, d) is called hyperconvex if

$$\bigcap_{\alpha \in \Gamma} B(x_\alpha, r_\alpha) \neq \emptyset$$

for any collection of closed balls $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Gamma}$ such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta,\ \alpha, \beta \in \Gamma$.

The space $C(S)$ of continuous real functions on a Stonian space with the "supremum" norm is hyperconvex and every hyperconvex real Banach space is $C(S)$ for some Stonian space. The standard examples of hyperconvex spaces are ℓ_∞, L_∞ and their unit balls.

The terminology is due to N. Aronszajn and P. Panitchpakdi [2] who proved that a hyperconvex space is a nonexpansive retract of any metric space in which it is isometrically embedded. J. Isbell [16] showed that every metric space is isometric to a subspace of a unique "minimal" hyperconvex space called the injective envelope. This notion was later rediscovered by A. Dress in [7] as the tight span in the context of optimal networks and phylogenetic analysis. For a deeper discussion of hyperconvex spaces we refer the reader to [9, 10, 11].

In 1979, R. Sine [23] and P. Soardi [24] showed independently that nonexpansive mappings (i.e., mappings which satisfy $d(Tx, Ty) \leq d(x, y),\ x, y \in M$) defined on a bounded hyperconvex space has fixed points. Since then, a number of fixed-point results in hyperconvex spaces were obtained of both topological and metric character. In particular, J. Baillon [3] showed that any intersection of hyperconvex spaces with a certain finite intersection

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property is a nonempty hyperconvex space. As a consequence, he proved that the set of common fixed points of a commuting family of nonexpansive mappings acting on a bounded hyperconvex space is a nonexpansive retract of M.

In this paper we focus on properties of fixed-point sets of uniformly Lipschitzian group actions on hyperconvex and λ-hyperconvex spaces. Uniformly Lipschitzian mappings, introduced in [12], are natural generalization of nonexpansive mappings. We say that a mapping $T : M \to M$ is uniformly L-Lipschitzian if $d(T^n x, T^n y) \leq Ld(x, y)$ for each $x, y \in M$ and $n \in \mathbb{N}$. For example, Lipschitzian periodic mappings are uniformly Lipschitzian. E. A. Lifschitz [20] proved that if C is a bounded, closed and convex subset of a Hilbert space and $L < \sqrt{2}$, then every uniformly L-Lipschitzian mapping $T : C \to C$ has a fixed point. From the geometric point of view, Hilbert and hyperconvex spaces are two extremes and yet there are some similarities between them. It was proved in [21] that every uniformly L-Lipschitzian mapping with $L < \sqrt{2}$ in a bounded hyperconvex space with the so-called property (P) has a fixed point. This result was later generalized in [6] but a question whether a counterpart of Lifshitz's theorem holds in every bounded hyperconvex space remains open.

We partially answer this question in Theorem 2.3 in the case of uniformly Lipschitzian group actions. Note that we assume the boundedness of orbits only, instead of the boundedness of a target space. Theorem 2.9 generalizes the result to the case of λ-hyperconvex spaces. As a consequence, we obtain the following rather surprising result, proved recently by U. Lang [19]: there exists a common fixed point for all surjective isometries acting on a bounded hyperconvex space. We also give a fixed point theorem for L-Lipschitzian involutions (Theorem 2.11). Note that M. S. Brodskiĭ and D. P. Mil'man [5] proved a similar statement for all surjective isometries acting on a weakly compact, convex subset of a Banach space with normal structure.

In Section 3 we give a qualitative complement to Theorem 2.3: if $\{T_a : a \in G\}$ is a group of uniformly L-Lipschitzian mappings on a hyperconvex space M with $L < \sqrt{2}$ and the orbits are bounded, then $\operatorname{Fix} G$ is a Hölder continuous retract of M.

We begin with some basic definitions and notation. Let (M, d) be a metric space. A semigroup $(S, \cdot)$ is said to act on M (from the left) if there is a map

$$\varphi : S \times M \to M$$

such that

$$\varphi(a, \varphi(b, x)) = \varphi(a \cdot b, x)$$

for all $a, b \in S$ and $x \in M$. If S has the identity element e, we further assume that $\varphi(e, x) = x,\ x \in M$.

(1) If $T : M \to M$ is a mapping, then $(\mathbb{N}, +)$ acts on M by

$$\varphi(0, x) = x,\ \varphi(n, x) = T^n x.$$

(2) If $T : M \to M$ is a bijection, then we get in a similar way an action of a group $(\mathbb{Z}, +)$ on M.

(3) In general, if a group G acts on M, then $\varphi(a,\cdot)$ is a bijection for every $a \in G$.
(4) If $G = \mathbb{Z}_n$, then $\varphi(1,\cdot)$ is a periodic mapping with a period n.

In this paper we study uniformly Lipschitzian group actions.

Definition 1.1. A group (or a semigroup) G is said to act uniformly L-Lipschitzly on a metric space M ($L > 0$) if

$$\forall a \in G\ \forall x, y \in M\ \ d(\varphi(a,x), \varphi(a,y)) \leq L d(x,y).$$

Remark 1.2. If $(\mathbb{N},+)$ acts on M, then we obtain the usual definition of a uniformly Lipschitzian mapping. If $G = \mathbb{Z}$ we recover the definition of a uniformly bi-Lipschitzian mapping.

From now on we shall write $\{T_a : a \in G\}$ for a representation of G as uniformly Lipschitzian mappings, i.e., $T_a x = \varphi(a,x), a \in G, x \in M$. Define a metric

$$d_G(x,y) = \sup_{a \in G} d(T_a x, T_a y).$$

Notice that the mappings T_a are nonexpansive in this metric and even isometries if G is a group. The metric d_G is equivalent to d since

$$d(x,y) \leq d_G(x,y) \leq L d(x,y).$$

It follows that a group (a semigroup, resp.) acts uniformly Lipschitzly iff it acts isometrically (nonexpansively, resp.) in an equivalent metric.

Recall that the orbit of the point $x \in M$ is the set

$$O(x) = \{T_a x : a \in G\}$$

and its diameter $\delta(x) = \sup_{a,b \in G} d(T_a x, T_b x)$. We say that the orbit is bounded if $\delta(x) < \infty$. The radius of the orbit $O(x)$ relative to a point $y \in M$ is defined by

$$r(y,x) = \sup_{a \in G} d(y, T_a x)$$

and the radius of $O(x)$ is given by

$$r(x) = \inf_{y \in M} r(y,x).$$

It is easy to see that

$$\frac{\delta(x)}{2} \leq r(x) \leq \delta(x).$$

Notice that if a group G acts on M, then the orbits are disjoint or equal, in other words they form the equivalence classes of this action.

Lemma 1.3. *If $\{T_a : a \in G\}$ is a group (or a semigroup) of uniformly Lipschitzian mappings, then all orbits are simultaneously bounded or unbounded.*

Proof. Assume that $\delta(x) < \infty$ for some $x \in M$. Then, for every $y \in M$,

$$\begin{aligned}\delta(y) &= \sup_{a,b \in G} d(T_a y, T_b y) \leq \sup_{a,b \in G} \left(d(T_a y, T_a x) + d(T_a x, T_b x) + d(T_b x, T_b y)\right) \\ &\leq 2Ld(x,y) + \delta(x) < \infty.\end{aligned}$$

□

The following definition is central for our work.

Definition 1.4. The center of the orbit of $x \in M$ is the set

$$C(x) = \{y \in M : r(y, x) = r(x)\}$$

and the center of $C(x)$ is defined by

$$CC(x) = \bigcap_{y \in C(x)} B(y, r(x)) \cap C(x).$$

2. Common fixed points

Definition 2.1. A point $x_0 \in M$ is a common fixed point for a group $\{T_a : a \in G\}$ if

$$\forall a \in G \; T_a x_0 = x_0.$$

The set of common fixed points is denoted by $\operatorname{Fix} G$.

Notice that if $\operatorname{Fix} G$ is nonempty and the group (semigroup) acts uniformly Lipschitzly, then all orbits are bounded, see Lemma 1.3, i.e., the boundedness of orbits is a necessary condition for the existence of common fixed points in this case.

In this section we prove theorems concerning the existence of common fixed points for uniformly Lipschitzian group actions on hyperconvex spaces. First recall some basic facts.

Definition 2.2. A metric space (M, d) is called hyperconvex if

$$\bigcap_{\alpha \in \Gamma} B(x_\alpha, r_\alpha) \neq \emptyset$$

for any collection of closed balls $\{B(x_\alpha, r_\alpha)\}_{\alpha \in \Gamma}$ such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta,\ \alpha, \beta \in \Gamma$.

It is not difficult to see that hyperconvex spaces are complete. We will use this fact several times. In hyperconvex spaces,

$$r(x) = \frac{\delta(x)}{2},$$

whenever the orbits are bounded. The sets $C(x), CC(x)$ are nonempty since $\operatorname{diam} C(x) \leq 2r(x)$. Furthermore,

$$C(x) = \bigcap_{a \in G} B(T_a x, r(x)).$$

Theorem 2.3. *If $\{T_a : a \in G\}$ is a group of uniformly L-Lipschitzian mappings on a hyperconvex space M with $L < \sqrt{2}$ and the orbits $O(x)$ are bounded, then* $\operatorname{Fix} G$ *is nonempty.*

Proof. Without loss of generality we can assume that $L \in \left(1, \sqrt{2}\right)$. Fix $x_1 \in M$ and select $x_2 \in CC(x_1)$. Then

$$\forall a, b \in G \; d(T_a x_2, T_b x_1) \leq L d(x_2, T_{a^{-1}b} x_1) \leq L r(x_1).$$

Now notice that from hyperconvexity, for every $a \in G$ there exists $y_a \in M$ such that

$$y_a \in \bigcap_{b \epsilon G} B\left(T_b x_1, r\left(x_1\right)\right) \cap B\left(T_a x_2, \left(L-1\right) r\left(x_1\right)\right).$$

Hence $y_a \in C\left(x_1\right)$ and

$$d\left(y_a, T_a x_2\right) \leq \left(L-1\right) r\left(x_1\right).$$

Then

$$d\left(x_2, T_a x_2\right) \leq d\left(x_2, y_a\right) + d\left(y_a, T_a x_2\right) \leq r\left(x_1\right) + \left(L-1\right) r\left(x_1\right) = Lr\left(x_1\right)$$

and it follows that

$$d\left(T_a x_2, T_b x_2\right) \leq Ld\left(x_2, T_{a^{-1}b} x_2\right) \leq L^2 r\left(x_1\right)$$

for every $a, b \in G$. Hence

$$\delta\left(x_2\right) \leq \frac{L^2}{2} \delta\left(x_1\right).$$

Next we select $x_3 \in CC\left(x_2\right)$ and estimate $\delta\left(x_3\right)$ in a similar way. We continue in this fashion obtaining recursively a sequence $\left(x_n\right)$ such that $\delta\left(x_{n+1}\right) \leq \frac{L^2}{2} \delta\left(x_n\right)$ and $\delta\left(x_{n+1}, x_n\right) = \frac{\delta(x_n)}{2}$. It follows that $\left(x_n\right)$ is a Cauchy sequence converging to a point $x_0 \in \operatorname{Fix} G$ since

$$\delta\left(x_0\right) \leq 2Ld\left(x_0, x_n\right) + \delta\left(x_n\right) \rightarrow 0.$$

□

In the above theorem we do not assume the boundedness of M but the boundedness of orbits, only. The following example of S. Prus (see [18, p. 412]) shows that in the case of semigroups the boundedness of orbits does not imply the existence of a fixed point even for nonexpansive mappings.

Example 2.4. Let $M = \ell_\infty$ and consider the semigroup generated by the mapping $T : \ell_\infty \to \ell_\infty$ defined by

$$T(x_n) = \left(1 + \mathrm{LIM}\ x_n, x_1, x_2, ...\right),$$

where LIM denotes a Banach limit. Then $\|T^n x\| \leq 1 + \|x\|$ and

$$\|T^n x - T^n y\| = \|x - y\|$$

for every $x, y \in \ell_\infty$. If $\bar{x} \in \ell_\infty$ satisfies $T\bar{x} = \bar{x}$, then

$$\bar{x}_1 = 1 + \mathrm{LIM}\ \bar{x}_n,\ \bar{x}_2 = \bar{x}_1,\ \bar{x}_3 = \bar{x}_2, ...$$

and hence LIM $\bar{x}_n = 1 +$LIM $\bar{x}_n$, a contradiction, which shows that $\operatorname{Fix} T = \emptyset$.

An interesting special case of Theorem 2.3, proved independently in [19] (see also [8]), is concerned with the group of all surjective isometries on a bounded hyperconvex space M.

Corollary 2.5. *If $\{T_a : a \in G\}$ is a group of isometries with bounded orbits $O\left(x\right)$ on a hyperconvex space M, then* $\operatorname{Fix} G$ *is nonempty. In particular, if M is bounded, there exists a common fixed point for all surjective isometries on M.*

If M is unbounded, it is not difficult to find two surjective isometries without a common fixed point, however the group which they generate has unbounded orbits.

Example 2.6. Consider the space $\mathbb{R}^2$ with the maximum norm and let T_a, T_b be rotations around two distinct points $a, b \in \mathbb{R}^2$ through the angle $\theta = \frac{\pi}{2}$. Each of them is clearly a surjective isometry with bounded orbits but the composition $T_a \circ T_b$ (and hence the whole group) has unbounded orbits.

Now we give an analogous theorem for λ-hyperconvex spaces.

Definition 2.7. A subset D of a metric space M is called admissible if it is the intersection of closed balls.

Definition 2.8. A metric space (M, d) is said to be λ-hyperconvex if for every non-empty admissible set D and for any family of closed balls $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Gamma}$ centered at $x_\alpha \in D, \alpha \in \Gamma$, such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta,\ \alpha, \beta \in \Gamma$, the intersection

$$D \cap \bigcap_{\alpha\in\Gamma} B\left(x_\alpha, \lambda r_\alpha\right) \neq \emptyset.$$

Like hyperconvex, λ-hyperconvex spaces are complete. It is known that M is hyperconvex iff it is 1-hyperconvex. If $\{T_a : a \in G\}$ is a group of mappings acting on a λ-hyperconvex space, we have the following estimations:

$$\frac{\delta\left(x\right)}{2} \leq r\left(x\right) \leq \lambda \frac{\delta\left(x\right)}{2}.$$

However, the center $C\left(x\right)$ may be empty and instead we will use the sets

$$A\left(x\right) = \bigcap_{a\in G} B\left(T_a x, \frac{\lambda\delta\left(x\right)}{2}\right),$$

$$AA\left(x\right) = A\left(x\right) \cap \bigcap_{y\in A(x)} B\left(y, \frac{\lambda^2\delta\left(x\right)}{2}\right)$$

which are nonempty by the definition of λ-hyperconvexity.

Theorem 2.9. *If $\{T_a : a \in G\}$ is a group of uniformly L-Lipschitzian mappings on a λ-hyperconvex space M with $L < \frac{\sqrt{2}}{\lambda}$ and the orbits $O\left(x\right)$ are bounded, then* $\operatorname{Fix} G$ *is nonempty.*

Proof. We can assume that $L \in \left(1, \frac{\sqrt{2}}{\lambda}\right)$. Fix $x_1 \in M$ and select $x_2 \in AA\left(x_1\right)$. Then

$$\forall a, b \in G\ d\left(T_a x_2, T_b x_1\right) \leq L d\left(x_2, T_{a^{-1}b} x_1\right) \leq \frac{L\lambda\delta\left(x_1\right)}{2}.$$

It follows from λ-hyperconvexity that for every $a \in G$ there exists $y_a \in M$ such that

$$y_a \in \bigcap_{b\in G} B\left(T_b x_1, \frac{\lambda\delta\left(x_1\right)}{2}\right) \cap B\left(T_a x_2, \frac{\left(L-1\right)\lambda^2\delta\left(x_1\right)}{2}\right).$$

Hence $y_a \in A(x_1)$ and

$$d(y_a, T_a x_2) \leq \frac{(L-1)\lambda^2 \delta(x_1)}{2}.$$

Then

$$d(x_2, T_a x_2) \leq d(x_2, y_a) + d(y_a, T_a x_2) \leq \frac{\lambda^2 \delta(x_1)}{2} + \frac{(L-1)\lambda^2 \delta(x_1)}{2} = \frac{L\lambda^2 \delta(x_1)}{2}$$

and hence

$$d(\varphi(a, x_2), \varphi(b, x_2)) \leq Ld(x_2, \varphi(b-a, x_2)) \leq \frac{L^2 \lambda^2 \delta(x_1)}{2}$$

for every $a, b \in G$, which gives

$$\delta(x_2) \leq \frac{L^2\lambda^2}{2} \delta(x_1).$$

Now we select $x_3 \in AA(x_2)$ and estimate $\delta(x_3)$ analogously. Thus we obtain recursively a sequence (x_n) such that

$$\delta(x_{n+1}) \leq \frac{L^2\lambda^2}{2} \delta(x_n),$$

$$d(x_{n+1}, x_n) \leq \frac{\lambda \delta(x_n)}{2} \leq \frac{\lambda}{2} \left(\frac{L^2\lambda^2}{2}\right)^{n-1} \delta(x_1).$$

It follows that (x_n) is a Cauchy sequence (since $\frac{L^2\lambda^2}{2} < 1$) which converges to a point $x_0 \in \operatorname{Fix} G$. $\square$

Corollary 2.10. *If $\{T_a : a \in G\}$ is a group of isometries on a λ-hyperconvex space M with $\lambda < \sqrt{2}$ and the orbits are bounded, then* $\operatorname{Fix} G$ *is nonempty. In particular, if M is bounded, there exists a common fixed point for all surjective isometries on M.*

A natural example of group actions (where $G = \mathbb{Z}_n$) are n-periodic mappings. Uniformly Lipschitzian n-periodic mappings were studied by W. A. Kirk in [17], where the results for a constant L satisfying $n^{-2}[(n-1)(n-2)L^2 + 2(n-1)L] < 1$ were obtained in any bounded, closed, convex subset of a Banach space (see also [14]). Here we obtain the results independent of n and for some metric spaces that are not necessarily bounded.

If a group $\mathbb{Z}_2$ acts on a metric space, the orbits consist of two elements (so they are always bounded). In this case $T^2 = I$ and such a mapping T is called an involution. In the following theorem we consider λ-hyperconvex spaces.

Theorem 2.11. *If T is an L-Lipschitzian involution in a λ-hyperconvex space with $L < \frac{2}{\lambda^2}$, then* $\operatorname{Fix} T \neq \emptyset$.

Proof. In the case of an involution,

$$\delta(x) = d(x, Tx),$$
$$A(x) = B\left(x, \frac{\lambda\delta(x)}{2}\right) \cap B\left(Tx, \frac{\lambda\delta(x)}{2}\right),$$
$$AA(x) = \bigcap_{y\in A(x)} B\left(y, \frac{\lambda^2\delta(x)}{2}\right) \cap A(x).$$

Fix $x_1 \in M$ and select $x_2 \in AA(x)$. Then

$$d(Tx_2, x_1) \le \frac{L\lambda\delta(x_1)}{2},$$

and

$$d(Tx_2, Tx_1) \le \frac{L\lambda\delta(x_1)}{2}.$$

It follows from the definition of λ-hyperconvexity that there exists y such that

$$y \in B\left(x_1, \frac{\lambda\delta(x_1)}{2}\right) \cap B\left(\varphi(x_1), \frac{\lambda\delta(x_1)}{2}\right) \cap B\left(\varphi(x_2), \frac{(L-1)\lambda^2\delta(x_1)}{2}\right).$$

Hence

$$d(x_2, Tx_2) \le d(x_2, y) + d(y, Tx_2) \le \frac{L\lambda^2\delta(x_1)}{2}.$$

As in the proof of Theorem 2.9 we have a sequence (x_n) such that

$$\delta(x_{n+1}) \le \frac{L\lambda^2}{2}\delta(x_n),$$
$$d(x_{n+1}, x_n) \le \frac{\lambda\delta(x_n)}{2} \le \frac{\lambda}{2}\left(\frac{L\lambda^2}{2}\right)^{n-1}\delta(x_1).$$

It is enough to put $L < \frac{2}{\lambda^2}$ to guarantee that (x_n) is a Cauchy sequence. □

Corollary 2.12. *Every L-Lipschitzian involution in a hyperconvex space with $L < 2$ has a fixed point.*

The above corollary should be compared with the result of K. Goebel and E. Złotkiewicz [13] who proved that if C is a closed and convex subset of a Banach space, then every L-Lipschitzian involution $T : C \to C$ with $L < 2$ has a fixed point. Our result is the analogue for hyperconvex metric spaces.

3. Retractions onto Fix G

It was proved in [22] (see also [3]) that the set of fixed points of a nonexpansive mapping in hyperconvex spaces is itself hyperconvex and hence is a nonexpansive retract of the domain. This is no longer true if a mapping is Lipschitzian with a constant $k > 1$.

Example 3.1. Let $a \in (0, 1)$ and put

$$f_a(t) = \begin{cases} (1+a)t + a, & t \in [-1, 0] \\ (1-a)t + a, & t \in (0, 1]. \end{cases}$$

Define a mapping $T : B_{\ell_\infty} \to B_{\ell_\infty}$ by

$$Tx = (f_a(x_1), f_a(x_2), ...) .$$

Then

$$\operatorname{Fix} T = \{x \in B : x_n \in \{0,1\} \text{ for every } n \in \mathbb{N}\} .$$

The Lipschitz constant of T equals $1 + a$ which is close to 1 for a close to 0. Furthermore, T is a bi-Lipschitz bijection and hence we can define a $\mathbb{Z}$-action on B_{ℓ_∞} which is Lipschitzian (but not uniformly Lipschitzian). Notice that $\operatorname{Fix} T$ is not a (continuous) retract of B_{ℓ_∞} since it consists of points of distance 2 apart.

We show that if a group action on a hyperconvex space M is L-uniformly Lipschitzian with $L < \sqrt{2}$, then the set of common fixed points is a Hölder continuous retract of M. We begin with a few lemmas concerned with the radius and the center of a subset K of a metric space M. Let

$$\begin{aligned} r(y,K) &= \sup_{x\in A} d(y,x), \\ r(K) &= \inf_{y\in M} \sup_{x\in K} d(y,x), \\ C(K) &= \left\{ y \in M : \sup_{x\in K} d(y,x) = r(K) \right\} \end{aligned}$$

denote the radius of K relative to y, the radius of K and the center of K relative to M, respectively. In general, $C(K)$ may be empty. The Hausdorff distance of bounded subsets K and L is defined by

$$D(K,L) = \max\{\sup_{x\in K} \inf_{y\in L} d(y,x), \sup_{x\in L} \inf_{y\in K} d(y,x)\}.$$

If

$$K_r = \bigcup_{x\in A} B(x,r),$$

then

$$D(K,L) = \inf\{r > 0 : K \subset L_r \wedge L \subset K_r\}.$$

Lemma 3.2. *For bounded subsets K, L of a metric space M,*

$$|r(K) - r(L)| \le D(K,L).$$

Proof. Fix $\varepsilon > 0$. There exist $p, q \in M$ such that

$$K \subset B(p, r(K) + \varepsilon), \ L \subset B(q, r(L) + \varepsilon)$$

since

$$\forall x \in K \ \exists y \in L \ d(x,y) \le D(K,L) + \varepsilon.$$

Hence

$$r(q,K) \le r(L) + \varepsilon + D(K,L) + \varepsilon$$

which gives

$$r(K) \le r(L) + D(K,L) + 2\varepsilon.$$

In a similar way we show that

$$r(L) \le r(K) + D(K,L) + 2\varepsilon.$$

This completes the proof since ε is arbitrary. $\square$

Lemma 3.3. *In hyperconvex spaces,*

$$D\left(C\left(K\right),C\left(L\right)\right)\leq D\left(K,L\right)+\left|r\left(K\right)-r\left(L\right)\right|.$$

Proof. Notice first that $C\left(K\right),C\left(L\right)$ are nonempty in a hyperconvex space. Fix $\varepsilon>0$ and $q\in C\left(L\right)$. From the definition of the Hausdorff metric,

$$\forall x\in K\ \exists y\in L\ d\left(x,y\right)\leq D\left(K,L\right)+\varepsilon$$

which gives

$$\forall x\in K\ d\left(x,q\right)\leq D\left(K,L\right)+r\left(L\right)+\varepsilon.$$

It follows that there exists $p\in C\left(K\right)$ such that

$$p\in\bigcap_{x\in K}B\left(x,r\left(K\right)\right)\cap B\left(q,D\left(K,L\right)+r\left(L\right)-r\left(K\right)+\varepsilon\right).$$

Hence

$$\forall q\in C\left(L\right)\ \ \exists p\in C\left(K\right)\ \ d\left(p,q\right)\leq D\left(K,L\right)+r\left(L\right)-r\left(K\right)+\varepsilon.$$

Analogously,

$$\forall p\in C\left(K\right)\ \ \exists q\in C\left(L\right)\ \ d\left(p,q\right)\leq D\left(K,L\right)+r\left(K\right)-r\left(L\right)+\varepsilon.$$

This completes the proof since ε is arbitrary. □

Corollary 3.4. *In hyperconvex spaces,*

$$D\left(C\left(K\right),C\left(L\right)\right)\leq 2D\left(K,L\right),$$

$$r\left(K\right)=r\left(L\right)\Rightarrow D\left(C\left(K\right),C\left(L\right)\right)=D\left(K,L\right).$$

Example 3.5. Consider the space $\mathbb{R}^2$ with the maximum norm and let $K=\left\{(0,0)\right\},L=\left\{(t,1),t\in[-1,1]\right\}$. We have $C\left(K\right)=\left\{(0,0)\right\},C\left(L\right)=\left\{(0,t),t\in[0,2]\right\}$. Then

$$D\left(C\left(K\right),C\left(L\right)\right)=2=2D\left(K,L\right)$$

which shows that the estimation in Corollary 3.4 is sharp.

Remark 3.6. In general spaces we have no similar estimation. Even in a Hilbert space, the distance of centers $D\left(C\left(K\right),C\left(L\right)\right)$ does not depend on the distance $D\left(K,L\right)$ in a Lipschitz way, see [25, Theorem 1], [1, Theorem 7]. The following example shows that there is no hope to extend Lemma 3.3 to λ-hyperconvex spaces.

Example 3.7. Let $M=\left\{z=e^{i\varphi},\varphi\in[0,2\pi)\right\}$ with the metric

$$d\left(z_1,z_2\right)=d\left(e^{i\varphi},e^{i\psi}\right)=\begin{cases}\left|\varphi-\psi\right|, & \text{if } \left|\varphi-\psi\right|\leq\pi,\\ 2\pi-\left|\varphi-\psi\right|, & \text{if } \left|\varphi-\psi\right|>\pi.\end{cases}$$

Note that this is a 2-hyperconvex space. Fix $\varepsilon>0$ and let $K=\{1,-1\}=\left\{e^0,e^{i\pi}\right\},L=\left\{e^{i(\pi-\varepsilon)},1\right\}$. Then $C\left(K\right)=\{i,-i\}=\left\{e^{i\frac{\pi}{2}},e^{-i\frac{\pi}{2}}\right\},C\left(L\right)=\left\{e^{i\frac{\pi-\varepsilon}{2}}\right\}$ and hence

$$D\left(C\left(K\right),C\left(L\right)\right)=\pi-\frac{\varepsilon}{2},$$

$$D\left(K,L\right)=\varepsilon.$$

Taking ε close to 0, it follows that there is no continuous dependence of $D\left(C\left(K\right),C\left(L\right)\right)$ on $D\left(K,L\right)$ in this case.

We conclude with proving a qualitative version of Theorem 2.3. The proof relies on the following two results. The first one is a special case of [15, Theorem 1]. Let $\mathcal{A}(M)$ denote the family of all nonempty admissible subsets of M.

Theorem 3.8. *Let M be a hyperconvex metric space, let S be any set, and let $T^* : S \to \mathcal{A}(M)$. Then there exists a mapping $T : S \to M$ for which $Tx \in T^*x$ for $x \in S$ and for which $d(Tx,Ty) \leq D(T^*x,T^*y)$ for each $x,y \in S$.*

Theorem 3.9 (see, e.g., [4, Prop. 1.10], [26, Lemma 2.2])**.** *Let (X,d) be a complete bounded metric space and let $f : X \to X$ be a k-Lipschitzian mapping. Suppose there exists $0<\gamma<1$ and $c>0$ such that $d(f^{n+1}(x),f^n(x)) \leq c\gamma^n$ for every $x \in X$. Then $Rx = \lim_{n\to\infty} f^n(x)$ is a Hölder continuous mapping.*

If we analyze the proof of Theorem 2.3, we see that the sets $C\left(x\right),CC\left(x\right)$ are admissible. Furthermore, if a group acts uniformly L-Lipschitzly on a hyperconvex space, then by Corollary 3.4,

$$\begin{aligned} D\left(O\left(x\right),O\left(y\right)\right) &\leq Ld\left(x,y\right), \\ D\left(C\left(x\right),C\left(y\right)\right) &\leq 2Ld\left(x,y\right), \\ D\left(CC\left(x\right),CC\left(y\right)\right) &\leq 4Ld\left(x,y\right) \end{aligned}$$

and we can use Theorem 3.8 to obtain a $4L$-Lipschitzian selection $f : M \to M$ such that $f(x) \in CC\left(x\right), x \in M$. This leads to the following result.

Theorem 3.10. *If $\{T_a : a \in G\}$ is a group of uniformly L-Lipschitzian mappings on a hyperconvex space M with $L<\sqrt{2}$ and the orbits are bounded, then $\operatorname{Fix} G$ is a Hölder continuous retract of M.*

Proof. Let $L \in \left(1,\sqrt{2}\right)$. The observation given above gives a $4L$-Lipschitzian mapping $f : M \to M$ such that $f\left(x\right) \in CC\left(x\right)$ for each $x \in M$. Fix $\bar{x} \in M$ and let $x_1 = f(\bar{x}) \in CC\left(\bar{x}\right)$. Now we can follow the proof of Theorem 2.3 to get

$$\delta\left(f(\bar{x})\right) \leq \frac{L^2}{2}\delta\left(\bar{x}\right).$$

Applying this argument again, we obtain recursively a sequence $\left(f^n(\bar{x})\right)$ such that $\delta\left(f^{n+1}(\bar{x})\right) \leq \frac{L^2}{2}\delta\left(f^n(\bar{x})\right)$ and $d\left(f^{n+1}(\bar{x}),f^n(\bar{x})\right) = \frac{\delta(f^n(\bar{x}))}{2}$. It follows that $\left(f^n(\bar{x})\right)$ is a Cauchy sequence and we can define a mapping

$$R\bar{x} = \lim_{n\to\infty} f^n(\bar{x})$$

for every $\bar{x} \in M$. It is not difficult to see that $T_aR\bar{x} = R\bar{x}$ for every $a \in G$ and $\bar{x} \in M$. Furthermore, $R\bar{x} = \bar{x}$ if $\bar{x} \in \operatorname{Fix} G$ since then $f(\bar{x}) = \bar{x}$. Using Theorem 3.9 it follows that $R : M \to \operatorname{Fix} G$ is a Hölder continuous retraction onto $\operatorname{Fix} G$. □

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